

PREDUALS OF JBW*-TRIPLES ARE 1-PLICHKO SPACES

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ABSTRACT. We prove that the predual, M_* , of a JBW*-triple M is a 1-Plichko space (i.e. it admits a countably 1-norming Markushevich basis or, equivalently, it has a commutative 1-projectional skeleton), and obtain a natural description of the Σ -subspace of M . This generalizes and improves similar results for von Neumann algebras and JBW*-algebras. Consequently, dual spaces of JB*-triples also are 1-Plichko spaces. We also show that M_* is weakly Lindelöf determined if and only if M is σ -finite if and only if M_* is weakly compactly generated. Moreover, contrary to the proof for JBW*-algebras, our proof dispenses with the use of elementary submodels theory.

1. INTRODUCTION

The topic of this paper concerns operator algebras, Jordan structures, and Banach space theory. The main goal is to prove that the predual of any JBW*-triple satisfies the remarkable Banach space feature called 1-Plichko property. The predual of a JBW*-triple can be viewed as a non-commutative and non-associative generalization of an L^1 space. In general such a space may be highly non-separable. Despite this fact, our result implies that the predual of a JBW*-triple admits a nice decomposition into separable subspaces and admits an appropriate Markushevich basis. More precisely, let X be a Banach space. A subspace $D \subset X^*$ is said to be a Σ -subspace of X^* if there is a linearly dense set $S \subset X$ such that

$$D = \{\phi \in X^* : \{m \in S : \phi(m) \neq 0\} \text{ is countable}\}.$$

The Banach space X is called *(r-)Plichko* if X^* admits a (r) -norming Σ -subspace, i.e. there exists a Σ -subspace D of X^* such that

$$\|x\| \leq r \sup\{|\phi(x)| : \phi \in D, \|\phi\| \leq 1\} \quad (x \in X)$$

(compare [39, 42]). The 1-Plichko property is equivalent to the fact that X has a countably 1-norming Markushevich basis [39, Lemma 4.19]. Another deep result [46, Theorem 27] says that X is a 1-Plichko space if and only if it admits a commutative 1-projectional skeleton. A commutative 1-projectional skeleton is a system $(P_\lambda)_{\lambda \in \Lambda}$ of norm one projections on X , where Λ is an up-directed set, fulfilling the following conditions:

- $P_\lambda X$ is separable for each λ and $X = \bigcup_{\lambda \in \Lambda} P_\lambda X$.

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- $P_\lambda P_\mu = P_\lambda$ whenever $\lambda \leq \mu$.
- $P_\lambda P_\mu = P_\mu P_\lambda$ for all λ and μ .
- if (λ_n) is an increasing net in Λ , it has a supremum, $\lambda \in \Lambda$, and $P_\lambda X = \overline{\bigcup_n P_{\lambda_n} X}$.

It easily follows that any 1-Plichko space enjoys the 1-separable complementation property saying that any separable subspace can be enlarged to a 1-complemented separable subspace. This property was established by U. Haagerup for preduals of von Neumann algebras with the help of results from modular theory of von Neumann algebras (see [30, Theorem IX.1]).

The category of 1-Plichko spaces involves many classes of spaces studied in Banach space theory. Let us recall that X is *weakly Lindelöf determined*, WLD in short, if X^* is a Σ -subspace of itself. X is called *weakly compactly generated* (WCG in short) if it contains a weakly compact subset whose linear span is dense in X . Obviously, every WLD space is 1-Plichko, and it follows from [1, Proposition 2] that every WCG space is WLD. Plichko and 1-Plichko spaces were formally introduced in [39, §4.2]. The notion was motivated by a series of papers where A.N. Plichko studied this property under equivalent reformulations (see [53, 54, 55, 56]). Although the term 1-Plichko is the most commonly used name for the spaces defined above, they have been also studied under different names. Namely, the class of those Banach spaces which are 1-Plichko is precisely the class termed \mathcal{V} by J. Orihuela in [49], which has been also studied by M. Valdivia in [61].

It has been proved by the third author of this note in [42] that many important spaces have 1-Plichko property, for example L^1 spaces for non-negative σ -finite measures, order-continuous Banach lattices, and $C(K)$ -spaces for abelian compact groups K . Moreover, the paper [42] contains the first result on non-commutative L^1 spaces showing that the predual of a semi-finite von Neumann algebra is 1-Plichko. Motivated by the latter, the first three authors of this paper prove in [4] that the predual of any von Neumann algebra is 1-Plichko. Moreover, they showed that the canonical 1-norming Σ -subspace is the space of all elements whose range projection is σ -finite. A generalization to JBW*-algebras appeared to be non-trivial. In [5] the same authors showed that the predual of any JBW*-algebra is 1-Plichko. The proof was quite different from that given in the setting of von Neumann algebras. The proof in the Jordan case was based on constructing a special projection skeleton with the help of the set theoretical tool of elementary submodels. Obviously, the question whether, as in the case of von Neumann algebra preduals [4], the result can be obtained without any use of submodels theory is a gap which is not fulfilled by the results in [5].

In the present paper we prove a further generalization of the above mentioned results by showing that all JBW*-triple preduals are 1-Plichko spaces. Our main result reads as follows.

Theorem 1.1. *The predual M_* of a JBW*-triple M is a 1-Plichko space. Moreover, M_* is weakly Lindelöf determined if and only if M is σ -finite. In this case M_* is even weakly compactly generated.*

The approach in this paper resembles more the one of [4] than the one of [5]. One reason for this has already been mentioned, in the present paper the proofs and arguments do not make use of the set theoretic tool of submodels. Moreover, the theory of JBW*-triples allows to connect the description of the Σ -subspace obtained

in [4] and to obtain a similar and satisfactory description for JBW*-triples (and hence also for JBW*-algebras), see Theorem 5.1. The key result for this approach is Proposition 4.3.

The relevant notions related to JBW*-triples are gathered in Section 2. Theorem 1.1 – in fact a more precise version of Theorem 1.1 – follows from Theorems 3.1 and 4.1 proved below.

Since the second dual of a JB*-triple is a JBW*-triple (see [11, Corollary 3.3.5]), the next result is a straightforward consequence of Theorem 1.1.

Corollary 1.2. *The dual space of a JB*-triple is a 1-Plichko space.* \square

We recall that a Banach space X has the (r) -separable complementation property if any separable subspace of X is contained in a (r) -complemented separable subspace of X (compare [30, page 92]). Since 1-Plichko spaces enjoy the 1-separable complementation property (which follows immediately from the characterization using a projectional skeleton formulated above), we also get the following result.

Corollary 1.3. *Preduals of JBW*-triples have the 1-separable complementation property.* \square

The above corollary is an extension of a result of U. Haagerup, who showed that the same statement holds for von Neumann algebra preduals (with different methods, see [30, Theorem IX.1]).

2. NOTATION AND PRELIMINARIES

In this section we recall basic notions and results on JBW*-triples and Plichko spaces. We also include some auxiliary results needed to prove our main results. For unexplained notation from Banach space theory we refer to [24]. The symbols \mathcal{B}_X and X^* will denote the closed unit ball and the dual of a Banach space X , respectively.

2.1. Elements of JBW*-triples. In [44], W. Kaup obtains an analytic-algebraic characterization of bounded symmetric domains in terms of the so-called JB*-triples, by showing that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple. Thanks to this result, JB*-triples offer a natural bridge to connect the infinite-dimensional holomorphy with functional analysis. We recall that a JB*-triple is a complex Banach space E equipped with a continuous ternary product $\{., ., .\}$, which is symmetric and bilinear in the outer variables and conjugate-linear in the middle one, satisfying the following properties:

- $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\}$ for all $a, b, c, x, y \in E$ (Jordan identity),
- the operator $x \mapsto \{a, a, x\}$ is a hermitian operator with nonnegative spectrum for each $a \in E$,
- $\|\{a, a, a\}\| = \|a\|^3$ for $a \in E$.

We recall that an operator $T \in B(E)$ is hermitian if and only if $\|\exp(irT)\| = 1$ for each $r \in \mathbb{R}$. For $a, b \in E$ we define a (linear) operator $L(a, b)$ on E by $L(a, b)(x) = \{a, b, x\}$, $x \in E$, and a conjugate-linear operator $Q(a, b)$ by $Q(a, b)(x) = \{a, x, b\}$. Given $a \in E$, the symbol $Q(a)$ will denote the operator on E defined by $Q(a) = Q(a, a)$.

Every C^* -algebra is a JB^* -triple with respect to the triple product given by $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$. The same triple product equips the space $B(H, K)$, of all bounded linear operators between complex Hilbert spaces H and K , with a structure of a JB^* -triple. Among the examples involving Jordan algebras, we can say that every JB^* -algebra is a JB^* -triple under the triple product $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$.

An element e in a JB^* -triple E is said to be a *tripotent* if $e = \{e, e, e\}$. If E is a von Neumann algebra viewed as a JBW^* -triple, then any projection is clearly a tripotent; in fact, an element of a von Neumann algebra is a tripotent if and only if it is a partial isometry.

For each tripotent $e \in E$, the mappings $P_i(e) : E \rightarrow E$ ($i = 0, 1, 2$) defined by

$$P_2(e) = L(e, e)(2L(e, e) - id_E), \quad P_1(e) = 4L(e, e)(id_E - L(e, e))$$

$$\text{and } P_0(e) = (id_E - L(e, e))(id_E - 2L(e, e))$$

are contractive linear projections (see [26, Corollary 1.2]), called the *Peirce projections* associated with e . It is known (cf. [11, p. 32]) that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$, and $P_0(e) = id_E - 2L(e, e) + Q(e)^2$. In case E is a von Neumann algebra, $e \in E$ a partial isometry, $q = e^*e$ the initial projection and $p = ee^*$ the final projection, we get

$$P_2(e)x = pxq, \quad P_1(e)x = px(1 - q) + (1 - p)xq \text{ and } P_0(e)x = (1 - p)x(1 - q).$$

If e is even a symmetric element (i.e. $e^* = e$) in the von Neumann algebra then we have $p = q$.

The range of $P_i(e)$ is the eigenspace, $E_i(e)$, of $L(e, e)$ corresponding to the eigenvalue $\frac{i}{2}$, and

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e)$$

is termed the *Peirce decomposition* of E relative to e . Clearly, $e \in E_2(e)$ and $P_k(e)(e) = 0$ for $k = 0, 1$. The following multiplication rules (known as Peirce rules or Peirce arithmetic) are satisfied:

$$(1) \quad \{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = \{0\},$$

$$(2) \quad \{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e),$$

where $E_{i-j+k}(e) = \{0\}$ whenever $i - j + k \notin \{0, 1, 2\}$ ([26] or [11, Theorem 1.2.44]). A tripotent e is called *complete* if $E_0(e) = \{0\}$. The complete tripotents of a JB^* -triple E are precisely the complex and the real extreme points of its closed unit ball (cf. [6, Lemma 4.1] and [43, Proposition 3.5] or [11, Theorem 3.2.3]). Therefore every JBW^* -triple contains an abundant collection of complete tripotents. If $E = E_2(e)$, or equivalently, if $\{e, e, x\} = x$ for all $x \in E$, we say that e is *unitary*.

For each tripotent e in a JB^* -triple, E , the Peirce-2 subspace $E_2(e)$ is a unital JB^* -algebra with unit e , product $a \circ_e b := \{a, e, b\}$ and involution $a^{*e} := \{e, a, e\}$ (cf. [11, §1.2 and Remark 3.2.2]). As we noticed above, every JB^* -algebra is a JB^* -triple with respect to the product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

Kaup's Banach-Stone theorem (see [44, Proposition 5.5]) assures that a surjective operator between JB^* -triples is an isometry if and only if it is a triple isomorphism.

Consequently, the triple product on $E_2(e)$ is uniquely determined by the expression

$$(3) \quad \{a, b, c\} = (a \circ_e b^{*e}) \circ_e c + (c \circ_e b^{*e}) \circ_e a - (a \circ_e c) \circ b^{*e},$$

for every $a, b, c \in E_2(e)$. Therefore, unital JB*-algebras are in one-to-one correspondence with JB*-triples admitting a unitary element (see also [10, 4.1.55]).

A JBW*-triple is a JB*-triple which is also a dual Banach space. Examples of JBW*-triples include von Neumann algebras and JBW*-algebras. Every JBW*-triple admits a unique isometric predual and its triple product is separately weak*-to-weak*-continuous ([3], [34], [11, Theorem 3.3.9]). Consequently, the Peirce projections associated with a tripotent in a JBW*-triple are weak*-to-weak*-continuous. Therefore, for each tripotent e in a JBW*-triple M , the Peirce subspace $M_2(e)$ is a JBW*-algebra. Unlike general JB*-triples, JBW*-triples admit a rather concrete representation which we recall in Section 2.4 below as it is the essential tool for proving our results.

Let a, b be elements in a JB*-triple E . Following standard terminology, we shall say that a and b are *algebraically orthogonal* or simply *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. If we consider a C*-algebra A as a JB*-triple, then two elements $a, b \in A$ are orthogonal in the C*-sense (i.e. $ab^* = b^*a = 0$) if and only if they are orthogonal in the triple sense. Orthogonality is a symmetric relation. By Peirce arithmetic it is immediate that all elements in $E_2(e)$ are orthogonal to all elements in $E_0(e)$, in particular, two tripotents $u, v \in E$ are orthogonal if and only if $u \in E_0(v)$ (and, by symmetry, if and only if $v \in E_0(u)$). We refer to [9, Lemma 1] for other useful characterizations of orthogonality and additional details not explained here.

The order in the partially ordered set of all tripotents in a JB*-triple E is defined as follows: Given two tripotents $e, u \in E$, we say that $e \leq u$ if $u - e$ is a tripotent which is orthogonal to e .

Lemma 2.1. ([26, Cor. 1.7], [11, Prop. 1.2.43]) *Let u, e be two tripotents in a JB*-triple E . The following assertions are equivalent.*

- (1) $e \leq u$.
- (2) $P_2(e)(u) = e$.
- (3) $\{u, e, u\} = e$.
- (4) $\{e, u, e\} = e$.
- (5) e is a projection (i.e. a self-adjoint idempotent) in the JB*-algebra $E_2(u)$.

For each norm-one functional φ in the predual, M_* , of a JBW*-triple M , there exists a unique tripotent $e \in M$ satisfying $\varphi = \varphi P_2(e)$ and $\varphi|_{M_2(e)}$ is a faithful normal state of the JBW*-algebra $M_2(e)$ (see [26, Proposition 2]). This unique tripotent e is called the *support tripotent* of φ , and will be denoted by $e(\varphi)$. It is explicitly shown in [26, part (b) in the proof of Proposition 2] that

$$(4) \quad \text{if } u \text{ is a tripotent in } M \text{ with } 1 = \|\varphi\| = \varphi(u), \text{ then } u \geq e(\varphi).$$

We recall that a subspace I of a JB*-triple E is called an *inner ideal*, provided $\{I, E, I\} \subseteq I$ (i.e., provided $\{a, b, c\} \in I$ whenever $a, c \in I$ and $b \in E$). Clearly, an inner ideal is a subtriple. Note that if e is a tripotent of a JBW*-triple M , then $M_2(e)$ is a weak*-closed subtriple of M ([11, Th. 1.2.47]). In a von Neumann algebra W (regarded as JBW*-triple) left and right ideals and sets of the form aWb

(with fixed $a, b \in W$) are inner ideals, whereas weak*-closed inner ideals are of the form pWq with projections $p, q \in W$ [16, Thm. 3.16].

Given an element x in a JB*-triple E the symbol E_x will denote the norm-closed subtriple of E generated by x , that is, the closed subspace generated by all odd powers $x^{[2n+1]}$, where $x^{[1]} = x$, $x^{[3]} = \{x, x, x\}$, and $x^{[2n+1]} = \{x, x, x^{[2n-1]}\}$ ($n \geq 2$) (compare, for example, [47, Sec. 3.3] or [11, Lemma 1.2.10]). It is known that there exists an isometric triple isomorphism $\Psi : E_x \rightarrow C_0(L)$ satisfying $\Psi(x)(t) = t$, for all t in L (compare [44, 1.15]), where $C_0(L)$ is the abelian C*-algebra of all complex-valued continuous functions on L vanishing at 0, L being a locally compact subset of $(0, \|x\|]$ satisfying that $L \cup \{0\}$ is compact. Thus, for any continuous function $f : L \cup \{0\} \rightarrow \mathbb{C}$ vanishing at 0, it is possible to give the usual meaning in the sense of functional calculus to $f(x) \in E_x$, via $f(x) = \Psi^{-1}(f)$.

For each norm-one element x in a JBW*-triple M , $r(x)$ will denote its *range tripotent*. We succinctly describe its definition. (More details are given for example in [51, Section 2.2] or in [15, comments before Lemma 3.1] or [8, §2]). For $x \in M$ with $\|x\| = 1$, the functions $t \rightarrow t^{\frac{1}{2n-1}}$ give rise to an increasing sequence $(x^{[\frac{1}{2n-1}]})$ which weak*-converges to $r(x)$ in M . The tripotent $r(x)$ is the smallest tripotent $e \in M$ satisfying that x is a positive element in the JBW*-algebra $M_2(e)$ (see, for example, [15, comments before Lemma 3.1] or [8, §2]). The inequality $x \leq r(x)$ holds in $M_2(r(x))$ for every norm-one element $x \in E$. For a non-zero element $z \in M$, the range tripotent of z , $r(z)$, is precisely the range tripotent of $\frac{z}{\|z\|}$, and we set $r(0) = 0$.

Let M be a JBW*-triple. We recall that a tripotent u in M is said to be *σ -finite* if u does not majorize an uncountable orthogonal subset of tripotents in M . Equivalently, u is a σ -finite tripotent in M if and only if there exists an element φ in M_* whose support tripotent $e(\varphi)$ coincides with u (cf. [19, Theorem 3.2]). Following standard notation, we shall say that M is σ -finite if every tripotent in M is σ -finite, equivalently, every orthogonal subset of tripotents in M is countable (cf. [19, Proposition 3.1]). It is also known that the sum of an orthogonal countable family of mutually orthogonal σ -finite tripotents in M is again a σ -finite tripotent (see [19, Theorem 3.4(i)]). It is further proved in [19, Theorem 3.4(ii)] that every tripotent in M is the supremum of a set of mutually orthogonal σ -finite tripotents in M .

When a von Neumann algebra W is regarded as a JBW*-triple, a projection p is σ -finite in the triple sense if and only if it is σ -finite or countably decomposable in the usual sense employed for von Neumann algebras (compare [58, Definition 2.1.8] or [60, Definition II.3.18]).

We will need the following properties of σ -finite tripotents which have been borrowed from [19].

Lemma 2.2. [19] *Let M be a JBW*-triple and let e be a tripotent of M . Then the following hold:*

- (i) $M_2(e)$ is a JBW*-subtriple of M and any tripotent $p \in M_2(e)$ is σ -finite in $M_2(e)$ if and only if it is σ -finite in M .
- (ii) e is σ -finite if and only if $M_2(e)$ is σ -finite.
- (iii) If e is σ -finite, then any tripotent in $M_2(e)$ is σ -finite in M .

Proof. Since $M_2(e)$ is a weak*-closed subtriple of M , assertion (i) follows from [19, Lemma 3.6(ii)]. Assertion (ii) follows from (i), [19, Theorem 4.4 (viii)-(ix)] and the fact that e is a complete tripotent in $M_2(e)$. Finally, assertion (iii) follows immediately from (i) and (ii). \square

For non explained notions concerning JB*-algebras and JB*-triples we refer to the monographs [10] and [11].

2.2. Contractive and bicontractive projections. One of the main properties enjoyed by any member E in the class of JB*-triples states that the image of a contractive projection $P : E \rightarrow E$ (where contractive means $\|P\| \leq 1$) is again a JB*-triple with triple product $\{x, y, z\}_P := P(\{x, y, z\})$ for x, y, z in $P(E)$ and

$$(5) \quad P\{a, x, b\} = P\{a, P(x), b\}, \quad a, b \in P(E), x \in E,$$

(see [45], [59] and [28]). It is further known that under these conditions $P(E)$ need not be, in general, a JB*-subtriple of E (compare [25, Example 1] or [45, Example 3]). But note that if $P(E)$ is known to be a subtriple then $\{\cdot, \cdot, \cdot\}_P$ coincides with the original triple product of E because in JB*-triples norm and triple product determine each other (see e.g. [11, Th. 3.1.7, 3.1.20]). Fortunately, more can be said about the JB*-triple structure of $P(E)$. It is known that $P(E)$ is isometrically isomorphic to a JB*-subtriple of E^{**} (see [29, Theorem 2]).

If $P : E \rightarrow E$ is even a bicontractive projection (where bicontractive means $\|P\| \leq 1$ and $\|I - P\| \leq 1$ – by I_V or simply I we denote the identity on a vector space V) on a JB*-triple, it satisfies a stronger property. Namely, $P(E)$ is then a JB*-subtriple of E , in particular (5) can be improved because the identities

$$(6) \quad P\{a, b, x\} = \{a, b, P(x)\} \quad \text{and} \quad P\{a, x, b\} = \{a, P(x), b\}$$

hold for $a, b \in P(E)$, $x \in E$ (cf. [29, §3]). It is further known that when P is bicontractive, there exists a surjective linear isometry (i.e. a triple automorphism) Θ on E of period 2 such that $P = \frac{1}{2}(I + \Theta)$ (see [29, Theorem 4]). Since, by another interesting property of JBW*-triples, every surjective linear isometry on a JBW*-triple is weak*-to-weak*-continuous (see [34, Proof of Theorem 3.2]) we have, as a consequence, that a bicontractive projection P on a JBW*-triple is weak*-to-weak*-continuous.

2.3. Von Neumann tensor products. We recall now some basic facts on von Neumann tensor products of von Neumann algebras. The theory has been essentially borrowed from [60, Chapter IV], and we refer to the latter monograph for additional results not commented here. Let $A \subset B(H)$ and $W \subset B(K)$ be von Neumann algebras. The algebraic tensor product $A \otimes W$ is canonically embedded into $B(H \otimes K)$, where $H \otimes K$ is the hilbertian tensor product of H and K (see [60, Definition IV.1.2]). The von Neumann algebra generated by the algebraic tensor product $A \otimes W$ is denoted $A \overline{\otimes} W$, and is called *the von Neumann tensor product* of A and W . Note that $A \overline{\otimes} W$ is the weak* closure of $A \otimes W$ in $B(H \otimes K)$ (see [60, §IV.5]).

If A is commutative, then the predual of $A \overline{\otimes} W$ is canonically identified with the projective tensor product of preduals, i.e.

$$(7) \quad (A \overline{\otimes} W)_* = A_* \widehat{\otimes}_\pi W_*.$$

This follows from [60, Theorem IV.7.17] (or rather [60, Section IV.7]). Furthermore, the special case of a separable W_* is treated in [58, Th. 1.22.13], while there is another approach via results on operator spaces: Results due to E.G. Effros and Z.J. Ruan show that equality (7) holds for any von Neumann algebra W , when the projective tensor product on the right-hand side is in the category of operator spaces ([22], [21, Theorem 7.2.4]). But if A is commutative, it carries the minimal operator-space structure by [21, Proposition 3.3.1] and hence the predual A_* carries the maximal structure by [21, (3.3.13) or (3.3.15) on p. 51], and hence by [21, (8.2.4) on p. 146] the projective tensor product in the category of operator spaces coincides with the projective tensor product in the Banach space sense.

Lemma 2.3. *Let A and W be von Neumann algebras with A commutative. Suppose $P : W \rightarrow W$ is a weak*-to-weak*-continuous contractive projection. Then the following holds:*

- (i) $P(W)$ is a JBW^* -triple with triple product $\{x, y, z\}_P := P(\{x, y, z\})$ for x, y, z in $P(W)$.
- (ii) $A \overline{\otimes} P(W)$, the weak*-closure of the algebraic tensor product $A \otimes P(W)$ in $A \overline{\otimes} W$, is the range of a weak*-to-weak*-continuous contractive projection Q on $A \overline{\otimes} W$.
- (iii) $A \overline{\otimes} P(W)$ is a JBW^* -triple with triple product $\{x, y, z\}_Q := Q(\{x, y, z\})$ for x, y, z in $A \overline{\otimes} P(W)$. Moreover,

$$(A \overline{\otimes} P(W))_* = A_* \widehat{\otimes}_\pi (P(W))_* = A_* \widehat{\otimes}_\pi P^*(W_*).$$

Proof. We know from Section 2.2 that statement (i) is satisfied.

Since P is weak*-to-weak* continuous, it is the dual map of a map $P_* : W_* \rightarrow W_*$. It is clear that P_* is a contractive projection on W_* . It follows from basic tensor product properties (cf. [12, 3.2] or [57, Proposition 2.3]) that $I \otimes P_*$ is a contractive projection on $A_* \widehat{\otimes}_\pi W_*$. Moreover, by [12, 3.8] or [57, Proposition 2.5] the norm on its range (which is the norm-closure of the algebraic tensor product $A_* \otimes P_*(W_*)$) is the projective norm coming from $A_* \widehat{\otimes}_\pi P_*(W_*)$.

Further, it is clear that the dual mapping $Q = (I \otimes P_*)^*$ is a weak*-to-weak*-continuous contractive projection on $(A_* \widehat{\otimes}_\pi W_*)^* = A \overline{\otimes} W$. Using the results commented in Section 2.2 we know that its range is a JBW^* -triple with the triple product defined in (iii). Since the range of Q is canonically identified with the dual of $A_* \widehat{\otimes}_\pi P_*(W_*)$, to complete the proof of (ii) and (iii) it is enough to show that the range of $(I \otimes P_*)^*$ is $A \overline{\otimes} P(W)$.

To show the desired equality we observe that the restriction of $(I \otimes P_*)^*$ to the algebraic tensor product $A \otimes W$ coincides with $I \otimes P$. Therefore the range of $(I \otimes P_*)^*$ contains $A \otimes P(W)$ and hence also its weak* closure $A \overline{\otimes} P(W)$. Conversely, since the unit ball $\mathcal{B}_{A \otimes W}$ is weak*-dense in $\mathcal{B}_{A \overline{\otimes} W}$ (for example by the Kaplansky density theorem), and $(I \otimes P_*)^*$ is weak*-to-weak*-continuous, $\mathcal{B}_{A \otimes W}$ is weak* dense in the unit ball of the range of $(I \otimes P_*)^*$ as well. This completes the proof. \square

Lemma 2.4. *Let A and W be von Neumann algebras with A commutative. Suppose $P : W \rightarrow W$ is a bicontractive projection. Then the following holds:*

- (i) $P(W)$ is a JBW^* -subtriple of W .
- (ii) $A \overline{\otimes} P(W)$, the weak*-closure of the algebraic tensor product $A \otimes P(W)$ in $A \overline{\otimes} W$, is the range of a bicontractive projection on $A \overline{\otimes} W$.

(iii) $A\overline{\otimes}P(W)$ is a JBW*-subtriple of $A\overline{\otimes}W$ and, moreover,

$$(A\overline{\otimes}P(W))_* = A_* \widehat{\otimes}_\pi (P(W))_* = A_* \widehat{\otimes}_\pi P^*(W_*).$$

Proof. By Section 2.2 we know that $P(W)$ is a JB*-subtriple of W and that P is weak*-to-weak*-continuous. Hence we can apply Lemma 2.3. Moreover, since P is even bicontractive, we get that P_* is bicontractive, and hence $I \otimes P_*$ and $Q = (I \otimes P_*)^*$ are bicontractive too. Finally, since Q is bicontractive, by Section 2.2 we get that $A\overline{\otimes}P(W)$ is a JBW*-subtriple of $A\overline{\otimes}W$. \square

2.4. Structure theory. In this subsection we recall an important structure result, due to G. Horn [35] and G. Horn and E. Neher [36], which allows us to represent every JBW*-triple in a concrete way. These results will be the main tool for proving that JBW*-triple preduals are 1-Plichko spaces.

We begin by recalling the definition of Cartan factors. There are six types of them (compare [11, Example 2.5.31]):

Type 1: A Cartan factor of type 1 coincides with a Banach space $B(H, K)$, of all bounded linear operators between two complex Hilbert spaces H and K , where the triple product is defined by $\{x, y, z\} = 2^{-1}(xy^*z + zy^*x)$. If $\dim H = \dim K$, then we can suppose $H = K$ and we get the von-Neumann algebra $B(H)$. If $\dim K < \dim H$, we may suppose that K is a closed subspace of H and then $B(H, K)$ is a JB*-subtriple of $B(H)$. Moreover, if p is the orthogonal projection of H onto K , then $x \mapsto px$ is a bicontractive projection of $B(H)$ onto $B(H, K)$. If $\dim K > \dim H$, we may suppose that H is a closed subspace of K , p the orthogonal projection of K onto H and then $x \mapsto xp$ is a bicontractive projection of $B(K)$ onto $B(H, K)$.

Types 2 and 3: Cartan factors of types 2 and 3 are the subtriples of $B(H)$ defined by $C_2 = \{x \in B(H) : x = -jx^*j\}$ and $C_3 = \{x \in B(H) : x = jx^*j\}$, respectively, where j is a conjugation (i.e. a conjugate-linear isometry of period 2) on H . If j is a conjugation on H , then there is an orthonormal basis $(e_\gamma)_{\gamma \in \Gamma}$ such that $j(\sum_{\gamma \in \Gamma} c_\gamma e_\gamma) = \sum_{\gamma \in \Gamma} \overline{c_\gamma} e_\gamma$. Each $x \in B(H)$ can be represented by a “matrix” $(x_{\gamma\delta})_{\gamma, \delta \in \Gamma}$. It is easy to check that the representing matrix of jx^*j is the transpose of the representing matrix of x . Hence, C_2 consists of operators with antisymmetric representing matrix and C_3 of operators with symmetric ones. Therefore, $P(x) = \frac{1}{2}(x^t + x)$ (where $x^t = jx^*j$ is the transpose of x with respect to the basis chosen above) is a bicontractive projection on $B(H)$ such that C_3 is the range of P , and C_2 is the range of $I - P$.

Type 4: A Cartan factor of type 4 (denoted by C_4) is a complex spin factor, that is, a complex Hilbert space (with inner product $\langle \cdot, \cdot \rangle$) provided with a conjugation $x \mapsto \overline{x}$, triple product

$$\{x, y, z\} = \langle x, y \rangle z + \langle z, y \rangle x - \langle x, \overline{z} \rangle \overline{y},$$

and norm given by $\|x\|^2 = \langle x, x \rangle + \sqrt{\langle x, x \rangle^2 - |\langle x, \overline{x} \rangle|^2}$. We point out that C_4 is isomorphic to a Hilbert space and hence, in particular, reflexive.

Types 5 and 6: All we need to know about Cartan factors of types 5 and 6 (also called *exceptional Cartan* factors) is that they are all finite dimensional.

Although H. Hanche-Olsen showed in [31, §5] that the standard method to define tensor products of JC-algebras (and JW*-triples) is, in general, hopeless, von Neumann tensor products can be applied in the representation theory of JBW*-triples.

Let A be a commutative von Neumann algebra and let C be a Cartan factor which can be realised as a JW*-subtriple of some $B(H)$. As before, the symbol $A\overline{\otimes}C$ will denote the weak*-closure of the algebraic tensor product $A \otimes C$ in the usual von Neumann tensor product $A\overline{\otimes}B(H)$ of A and $B(H)$. This applies to Cartan factors of types 1–4 (this is obvious for Cartan factors of types 1–3, the case of type 4 Cartan factors follows from [32, Theorem 6.2.3]).

The above construction does not cover Cartan factors of types 5 and 6. When C is an exceptional Cartan factor, $A\overline{\otimes}C$ will denote the injective tensor product of A and C , which can be identified with the space $C(\Omega, C)$, of all continuous functions on Ω with values in C endowed with the pointwise operations and the supremum norm, where Ω denotes the spectrum of A (cf. [57, p. 49]). We observe that if C is a finite dimensional Cartan factor which can be realised as a JW*-subtriple of some $B(H)$ both definitions above give the same object (cf. [60, Theorem IV.4.14]).

The structure theory settled by G. Horn and E. Neher [35, (1.7)], [36, (1.20)] proves that every JBW*-triple M writes (uniquely up to triple isomorphisms) in the form

$$(8) \quad M = \left(\bigoplus_{j \in \mathcal{J}} A_j \overline{\otimes} C_j \right)_{\ell_\infty} \oplus_{\ell_\infty} H(W, \alpha) \oplus_{\ell_\infty} pV,$$

where each A_j is a commutative von Neumann algebra, each C_j is a Cartan factor, W and V are continuous von Neumann algebras, p is a projection in V , α is a linear involution on W commuting with $*$, that is, a linear $*$ -antiautomorphism of period 2 on W , and $H(W, \alpha) = \{x \in W : \alpha(x) = x\}$.

2.5. Some facts on Plichko spaces. The following lemma sums up several basic properties of Σ -subspaces.

Lemma 2.5. *Let X be a Banach space and $S \subset X^*$ a Σ -subspace. Then the following hold:*

- (i) *S is weak*-countably closed. That is, $\overline{C}^{w*} \subset S$ whenever $C \subset S$ is countable. In particular, S is weak*-sequentially closed and norm-closed.*
- (ii) *Bounded subsets of S are weak*-Fréchet Urysohn. That is, given $A \subset S$ bounded and $x^* \in S$ such that $x^* \in \overline{A}^{w*}$, then there is a sequence (x_n^*) in A weak*-converging to x^* .*
- (iii) *Let $S' \subset X^*$ be any other subspace satisfying (i) and (ii). If $S \cap S'$ is 1-norming, then $S = S'$.*
- (iv) *If X is WLD, then X^* is the only norming Σ -subspace of X^* .*
- (v) *If S is 1-norming, then for any $x \in X$ there is $x^* \in S$ of norm one such that $x^*(x) = \|x\|$.*

Proof. Assertion (i) follows from the very definition of a Σ -subspace, assertion (ii) follows from [39, Lemma 1.6]. Assertion (iii) is an easy consequence of (i) and (ii) and follows from [40, Lemma 2] (in fact in the just quoted lemma it is assumed that S' is a Σ -subspace as well, but the proof uses only properties (i) and (ii)). Assertion (iv) follows immediately from (iii) and (v) is an easy consequence of (i). \square

We will also need the following easy lemma on quotients of 1-Plichko spaces.

Lemma 2.6. *Let X be a 1-Plichko Banach space, and let $S \subset X^*$ be a 1-norming Σ -subspace. Suppose that $Z \subset X^*$ is a weak*-closed subspace such that $S \cap \mathcal{B}_Z$ is weak* dense in \mathcal{B}_Z . Then $S \cap Z$ is a 1-norming Σ -subspace of $Z = (X/Z_\perp)^*$.*

Proof. Since Z is a weak*-closed subspace of the dual space X^* , it is canonically isometrically identified with $(X/Z_\perp)^*$. Further, by the assumptions it is clear that $S \cap Z$ is a 1-norming subspace of Z . It remains to show it is a Σ -subspace.

To do that, fix a linearly dense set $A \subset X$ such that

$$S = \{x^* \in X^* : \{x \in A : x^*(x) \neq 0\} \text{ is countable}\}.$$

Let \tilde{A} be the image of A in X/Z_\perp by the canonical quotient mapping. It is clear that \tilde{A} is linearly dense. Let

$$\tilde{S} = \{x^* \in Z = (X/Z_\perp)^* : \{x \in \tilde{A} : x^*(x) \neq 0\} \text{ is countable}\}$$

be the Σ -subspace induced by \tilde{A} . It is easy to check that $S \cap Z \subset \tilde{S}$. It follows from Lemma 2.5(iii) that $S \cap Z = \tilde{S}$, which completes the proof. \square

3. PREDUALS OF σ -FINITE JBW*-TRIPLES

The aim of this section is to prove the following result.

Theorem 3.1. *The predual of any σ -finite JBW*-triple is weakly compactly generated, in fact even Hilbert generated.*

Recall that a Banach space X is said to be *Hilbert-generated* if there is a Hilbert space H and a bounded linear mapping $T : H \rightarrow X$ with dense range. It is clear that any Hilbert-generated Banach space is weakly compactly generated (the generating weakly compact set is precisely $T(\mathcal{B}_H)$).

Theorem 3.1 above follows from the following stronger statement, which is a JBW*-triple analogue of [4, Lemma 3.3] for von Neumann algebras and of [5, Proposition 3.7] for JBW*-algebras.

Proposition 3.2. *Let e be a σ -finite tripotent in a JBW*-triple M . Then the predual of the space $M_2(e) \oplus M_1(e)$ (i.e. $(P_2(e) + P_1(e))^*(M_*)$) is Hilbert-generated.*

To see that Theorem 3.1 follows from the above proposition it is enough to use the fact that any JBW*-triple contains an abundant set of complete tripotents. In particular, any σ -finite JBW*-triple M contains a σ -finite complete tripotent $e \in M$ such that $M = M_2(e) \oplus M_1(e)$. Hence Proposition 3.2 entails Theorem 3.1.

Next let us focus on the proof of Proposition 3.2. Similarly as in the case of von Neumann algebras and JBW*-algebras it will be done by introducing a canonical (semi)definite inner product. In [2, Proposition 1.2], Barton and Friedman showed that given an element φ in the dual of a JB*-triple E and an element $z \in E$ such that $\varphi(z) = \|\varphi\| = \|z\| = 1$, the map $E \times E \ni (x, y) \mapsto \langle x, y \rangle_\varphi := \varphi\{x, y, z\}$ defines a hermitian semi-positive sesquilinear form with the associated pre-hilbertian seminorm $\|x\|_\varphi := (\varphi\{x, x, z\})^{1/2}$ on M and is independent of z .

We shall need the following technical lemma borrowed from [19, Lemma 4.1]:

Lemma 3.3. *Let M be a JBW*-triple, let $\varphi \in M_*$ be of norm one and let $e = e(\varphi) \in M$ be its support tripotent. Then the annihilator of the pre-hilbertian seminorm $\|\cdot\|_\varphi$ is precisely $M_0(e)$, that is,*

$$(9) \quad \{x \in M : \|x\|_\varphi = 0\} = M_0(e).$$

In particular, the restriction of $\|\cdot\|_\varphi$ to $M_2(e) \oplus M_1(e)$ is a pre-hilbertian norm and the restriction of $\langle \cdot, \cdot \rangle_\varphi$ to $M_2(e) \oplus M_1(e)$ is an inner product.

Proof. The first statement is proved in [19, Lemma 4.1], the positive definiteness of $\|\cdot\|_\varphi$ and of $\langle \cdot, \cdot \rangle_\varphi$ on $M_2(e) \oplus M_1(e)$ follows immediately (see also [26, Lemma 1.5], [50]). \square

Now we are ready to prove the main proposition of this section:

Proof of Proposition 3.2. Since e is a σ -finite tripotent there exists a norm-one normal functional $\varphi \in M_*$ such that $e = e(\varphi)$ is the support tripotent of φ . Denote by h_φ the pre-hilbertian space $M_2(e) \oplus M_1(e)$ equipped with the inner product $\langle \cdot, \cdot \rangle_\varphi = \varphi\{\cdot, \cdot, e\}$, and write H_φ for its completion. Let us first consider $\tilde{\Phi}(a)$ defined by $x \mapsto \langle x, a \rangle_\varphi$ for $a \in h_\varphi$, $x \in M$. By the Cauchy-Schwarz inequality we have

$$|\tilde{\Phi}(a)(x)| = |\langle x, a \rangle_\varphi| \leq \|x\|_\varphi \|a\|_\varphi \leq \|x\| \|a\|_\varphi$$

which, together with the separate w^* -continuity of the triple product, shows that $\tilde{\Phi}$ is a well-defined conjugate-linear contractive map from h_φ to M_* .

In order to see that the range of $\tilde{\Phi}$ is contained in $(M_2(e) \oplus M_1(e))_* = (P_2^*(e) + P_1^*(e))(M_*)$, let us observe that for any $a \in h_\varphi$ and $y \in M_0(e)$, we have $\|y\|_\varphi = 0$ by Lemma 3.3, and hence $\tilde{\Phi}(a)(y) = 0$.

Thus, by density of h_φ in H_φ , $\tilde{\Phi} = (P_2^*(e) + P_1^*(e))\tilde{\Phi}$ gives rise to a conjugate-linear continuous map $\Phi : H_\varphi \rightarrow (M_2(e) \oplus M_1(e))_*$.

We shall finally prove that Φ has norm-dense range. Suppose $z \in M_2(e) \oplus M_1(e)$ satisfies $\Phi(a)(z) = 0$ for every $a \in h_\varphi$. In particular, $0 = \Phi(z)(z) = \|z\|_\varphi^2$ and thus, by Lemma 3.3, $z = 0$. By the Hahn-Banach theorem, Φ has dense range. If we replace the map Φ by Φj , where j is a conjugation on H_φ , then we have a linear mapping. \square

4. THE CASE OF GENERAL JBW*-TRIPLES

In this section we state and prove Theorem 4.1, which gives a more precise version of the first part of Theorem 1.1.

To provide a precise formulation we introduce one more notation. For a JBW* triple M we define the set

$$M_\sigma = \{x \in M : \text{there is a } \sigma\text{-finite tripotent } e \in M \text{ such that } P_2(e)x = x\}$$

and note that

$$\begin{aligned} M_\sigma &= \{x \in M : \text{there is a } \sigma\text{-finite tripotent } e \in M \text{ such that } \{e, e, x\} = x\} \\ &= \{x \in M : r(x) \text{ is a } \sigma\text{-finite tripotent}\}. \end{aligned}$$

Indeed, the first equality follows from the fact that the range of $P_2(e)$ is the eigenspace of $L(e, e)$ corresponding to the eigenvalue 1. Let us show the second equality. The inclusion ' \supset ' is obvious. To show the converse inclusion, let $x \in M_\sigma$. Fix a σ -finite tripotent $e \in M$ with $x = P_2(e)x$, i.e., $x \in M_2(e)$. Since $M_2(e)$ is a JBW*-subtriple of M and $r(x)$ belongs to the JBW*-subtriple generated by x , we have $r(x) \in M_2(e)$ and so $r(x)$ is σ -finite by Lemma 2.2.

We mention the easy but useful fact that M_σ is 1-norming in M . To see this we simply observe that M_σ contains all σ -finite tripotents of M , or equivalently, all support tripotents of functionals in M_* .

Theorem 4.1. *The predual space of a JBW*-triple M is a 1-Plichko space. Moreover,*

$$(10) \quad M_\sigma \text{ is a 1-norming } \Sigma\text{-subspace of } M = (M_*)^*.$$

In particular, M_ is weakly Lindelöf determined if and only if M is σ -finite.*

It is not obvious that M_σ is a subspace, but this will follow by the proof of Theorem 4.1; it will be reproved a second time in Theorem 5.1.

The ‘in particular’ part of the theorem is an immediate consequence of the first statements of the theorem. Indeed, M is σ -finite if and only if $M = M_\sigma$ (cf. Lemma 2.2). Hence, if M is σ -finite, then M_* is WLD by the first statement. Conversely, if M_* is WLD, then by the first part of the theorem together with Lemma 2.5(iv) we get $M = M_\sigma$, hence M is σ -finite. Thus, it is enough to prove (10). This will be done in the rest of this section by using results in [4] and the decomposition (8).

The following proposition is almost immediate from the main results of [4].

Proposition 4.2. *The statement of Theorem 4.1 holds for von Neumann algebras.*

Proof. It is enough to show (10) in case M is a von Neumann algebra. In view of [4, Proposition 4.1], to this end it is enough to show that

$$M_\sigma = \{x \in M : x = qxq \text{ for a } \sigma\text{-finite projection } q \in M\}.$$

Let x be in the set on the right-hand side. Fix a σ -finite projection $q \in M$ with $x = qxq$. Then q is a σ -finite tripotent and $\{q, q, x\} = \frac{1}{2}(qx + xq) = qxq = x$. Hence $x \in M_\sigma$.

Conversely, let $x \in M_\sigma$ and let $u \in M$ be a σ -finite triponent with $x = P_2(u)x$. Since M is a von Neumann algebra, u is a partial isometry and hence $P_2(u)x = pxq$, where $p = uu^*$ is the final projection and $q = u^*u$ is the initial projection. Then p is a σ -finite projection. Indeed, suppose that $(r_\gamma)_{\gamma \in \Gamma}$ is an uncountable family of pairwise orthogonal projections smaller than p . Then it is easy to check that $(r_\gamma u)_{\gamma \in \Gamma}$ is an uncountable family of pairwise orthogonal tripotents smaller than u . Similarly we get that q is σ -finite. Hence their supremum $r = p \vee q$ is σ -finite as well ([19, Theorem 3.4] or [38, Exercice 5.7.45]) and satisfies $x = rxr$. Thus x belongs to the set on the right-hand side and the proof is complete. \square

Proposition 4.3. *Let $P : M \rightarrow M$ be a bicontractive projection on a JBW*-triple, let $N = P(M)$, and let e be a tripotent in N . Then e is σ -finite in N if and only if e is σ -finite in M , that is, $N_\sigma = N \cap M_\sigma$.*

Proof. The ‘if’ implication is clear. Let e be a σ -finite tripotent in N . By [19, Theorem 3.2] there exists a norm-one functional $\phi \in N_*$ whose support tripotent in N is e . Let us define $\psi = P^*(\phi) = \phi P \in M_*$. Clearly $\|\psi\| = 1$. We shall prove that e is the support tripotent of ψ in M , and hence e is σ -finite in M ([19, Theorem 3.2]). Let u be the support tripotent of ψ in M . From $\psi(e) = \phi(e) = 1 = \|\psi\|$ we get $e \geq u$ (compare [26, part (b) in the proof of Proposition 2]).

We set $u_1 = P(u)$ and $u_2 = u - u_1$. Since $e \geq u$ in M , we deduce that $\{e, u, e\} = u = \{e, e, u\}$ ($e - u \in M_0(u)$ and Peirce rules). Hence, $u_1 = P(u) = \{e, Pu, e\} =$

$\{e, u_1, e\}$ and $u_1 = \{e, e, u_1\}$ by (6). It follows that $u_1 = \{e, u_1, e\} \in M_2(e)$ and that $u_1 = \{e, u_1, e\} = u_1^{*e}$ is a hermitian element in the closed unit ball of the JBW*-algebra $N_2(e)$. As e is the unit in this algebra and u_1 is a hermitian element of norm less than one, we see that $e - u_1$ is a positive element in the JBW*-algebra $N_2(e)$. The condition

$$\phi(e) = 1 = \psi(u) = \phi P(u) = \phi(u_1)$$

implies, by the faithfulness of $\phi|_{N_2(e)}$, that $u_1 = e$.

It follows from the above that $u_2 = \{e, e, u\} - \{e, e, u_1\} = \{e, e, u_2\}$ and similarly $u_2 = \{e, u_2, e\}$. These identities combined with the fact that $u = e + u_2$ is a tripotent (that is, $\{e + u_2, e + u_2, e + u_2\} = e + u_2$) yield

$$e + u_2 = e + 2\{u_2, u_2, e\} + \{u_2, e, u_2\} + 3u_2 + \{u_2, u_2, u_2\}.$$

After applying the bicontractive projection $I - P$ in both terms of the last equality we get $-2u_2 = \{u_2, u_2, u_2\}$. Now $2\|u_2\| = \|\{u_2, u_2, u_2\}\| = \|u_2\|^3$ implies either $u_2 = 0$ or $\|u_2\|^2 = 2$. The latter is not possible because $\|u_2\| \leq 1$ by the fact that $u_2 = (I - P)u$ and $I - P$ is a contraction. Thus $u_2 = 0$, and hence $e = u$, which proves the first statement.

For the last identity we observe that for every element $x \in N$, its range tripotent $r(x)$ (in N or in M) lies in N . Suppose x is an element in N whose range tripotent is σ -finite in N . We deduce from the first statement that $r(x)$ is also σ -finite in M , and hence $N_\sigma \subseteq M_\sigma$. The inclusion $N_\sigma \supseteq M_\sigma \cap N$ is clear. \square

By combining Proposition 4.2, Proposition 4.3, and Lemma 2.6 we get the following proposition.

Proposition 4.4. *Let $P : W \rightarrow W$ be a bicontractive projection on a von Neumann algebra W , let $M = P(W)$. Then M_* is a 1-Plichko space. Furthermore, M_σ is a 1-norming Σ -subspace of M .*

Now we are ready to prove the validity of (10) for most of the summands from the representation (8):

Proposition 4.5. *Let M be a JBW*-triple of one of the following forms:*

- (a) $M = A \overline{\otimes} C$, where A is a commutative von Neumann algebra and C is a Cartan factor of type 1, 2 or 3.
- (b) $M = H(W, \alpha)$, where W is a von Neumann algebra and α is a linear involution on W commuting with $*$.
- (c) $M = pV$, where V is a von Neumann algebra and $p \in V$ is a projection.

Then M_σ is a 1-norming Σ -subspace of $M = (M_)^*$.*

Proof. We will apply Proposition 4.4. To do that it is enough to show that M is the range of a bicontractive projection on a von Neumann algebra.

(a) If C is a Cartan factor of type 1, 2 or 3, then C is the range of a bicontractive projection on a certain von Neumann algebra W , as it was previously observed after the definitions of the respective Cartan factors. The desired bicontractive projection on $A \overline{\otimes} W$ is finally given by Lemma 2.4.

(b) A bicontractive projection on W is given by $x \mapsto \frac{1}{2}(x + \alpha(x))$.

(c) The mapping $x \mapsto px$ defines a bicontractive projection on V . \square

The remaining summands from (8) are covered by the following theorem, which we formulate in a more abstract setting of Banach spaces.

Theorem 4.6. *Let (Ω, Σ, μ) be a measure space with a non-negative semifinite measure, and let E be a reflexive Banach space. Then the space $L^1(\mu, E)$ of Bochner-integrable functions is 1-Plichko. Furthermore, $L^1(\mu, E)$ is weakly Lindelöf determined if and only if μ is σ -finite, in the latter case it is even weakly compactly generated.*

More precisely, there is a family of finite measures $(\mu_\gamma)_{\gamma \in \Gamma}$ such that $L^1(\mu, E)$ is isometric to

$$\left(\bigoplus_{\gamma \in \Gamma} L^1(\mu_\gamma, E) \right)_{\ell_1}$$

and

$$S = \left\{ f = (f_\gamma)_{\gamma \in \Gamma} \in \left(\bigoplus_{\gamma \in \Gamma} L^\infty(\mu_\gamma, E) \right)_{\ell_\infty} : \{\gamma \in \Gamma : f_\gamma \neq 0\} \text{ is countable} \right\}$$

is a 1-norming Σ -subspace of $(L^1(\mu, E))^* = \left(\bigoplus_{\gamma \in \Gamma} L^\infty(\mu_\gamma, E) \right)_{\ell_\infty}$.

Proposition 4.7. *Let μ be a finite measure, and let E be a reflexive Banach space. Then $L^1(\mu, E)$ is weakly compactly generated.*

Proof. The proof is done similarly as in the scalar case (cf. [42, Theorem 5.1]). Let us consider the identity mapping $T : L^2(\mu, E) \rightarrow L^1(\mu, E)$. By the Cauchy-Schwarz inequality we get $\|T\| \leq \sqrt{\|\mu\|}$, hence T is a bounded linear operator. Moreover, the range of T is dense, since countably valued functions in $L^1(\mu, E)$ are dense in the latter space. Finally, $L^2(\mu, E)$ is reflexive because E and E^* have Radon-Nikodým property (see [13, Theorem IV.1.1]). Thus, $L^1(\mu, E)$ is indeed weakly compactly generated. \square

Remark: Note that if E is isomorphic to a Hilbert space, then we can even conclude that $L^1(\mu, E)$ is Hilbert generated, since in this case $L^2(\mu, E)$ is also isomorphic to a Hilbert space. Indeed, if E is even isometric to a Hilbert space, the norm on $L^2(\mu, E)$ is induced by the scalar product

$$\langle f, g \rangle = \int \langle f(\omega), g(\omega) \rangle d\mu(\omega).$$

Proof of Theorem 4.6. We imitate the proof of [42, Theorem 5.1]. Let $\mathcal{B} \subset \Sigma$ be a maximal family with the following properties:

- $0 < \mu(B) < +\infty$ for each $B \in \mathcal{B}$;
- $\mu(B_1 \cap B_2) = 0$ for each $B_1, B_2 \in \mathcal{B}$ distinct.

The existence of such a family follows immediately from Zorn's lemma.

Take any separable-valued Σ -measurable function $f : \Omega \rightarrow E$. Then clearly

$$\int \|f(\omega)\| d\mu(\omega) = \sum_{B \in \mathcal{B}} \int_B \|f(\omega)\| d\mu(\omega).$$

Therefore $L^1(\mu, E)$ is isometric to the ℓ_1 -sum of spaces $L^1(\mu|_B, E)$, $B \in \mathcal{B}$. Since $\mu|_B$ is finite for each $B \in \mathcal{B}$, $L^1(\mu|_B, E)$ is weakly compactly generated (and hence weakly Lindelöf determined) by the previous Proposition 4.7. Further, it is clear

that the dual of $L^1(\mu, E)$ is canonically isometric to the ℓ_∞ -sum of the family $\{(L^1(\mu|_B, E))^* : B \in \mathcal{B}\}$. More concretely, since E is reflexive, by [13, Theorem IV.1.1] we have $(L^1(\mu|_B, E))^* = L^\infty(\mu|_B, E^*)$ for each $B \in \mathcal{B}$, and hence

$$L^1(\mu, E)^* = \left(\bigoplus_{B \in \mathcal{B}} L^\infty(\mu|_B, E^*) \right)_{\ell_\infty}.$$

Finally, it follows from [39, Lemma 4.34] that

$$S = \left\{ (f_B)_{B \in \mathcal{B}} \in \left(\bigoplus_{B \in \mathcal{B}} L^\infty(\mu|_B, E^*) \right)_{\ell_\infty} : \{B \in \mathcal{B}; f_B \neq 0\} \text{ is countable} \right\}$$

is a 1-norming Σ -subspace of $(L^1(\mu, E))^*$.

To prove the last statement, it is enough to observe that μ is σ -finite if and only if \mathcal{B} is countable, that a countable ℓ_1 -sum of weakly compactly generated spaces is again weakly compactly generated and that an uncountable ℓ_1 -sum of nontrivial spaces contains $\ell_1(\omega_1)$ and hence is not weakly Lindelöf determined. (Recall that WLD property passes to subspaces.) \square

Proposition 4.8. *Let A be a commutative von Neumann algebra and C a Cartan factor. Then $(A \bar{\otimes} C)_* = A_* \hat{\otimes}_\pi C_*$.*

Proof. If C is a Cartan factor of type 1, 2 or 3, then C is the range of a bicontractive projection on a von Neumann algebra and hence the equality follows from Lemma 2.4.

If C is a type 4 Cartan factor, it follows from [23, Lemma 2.3] that C is the range of a (unital positive) contractive projection $P : B(H) \rightarrow B(H)$ where H is an appropriate Hilbert space. The mapping $P^{**} : B(H)^{**} \rightarrow B(H)^{**}$ is a weak*-to-weak*-continuous contractive projection on the von Neumann algebra $B(H)^{**}$ whose range is C by (Goldstine's theorem and) reflexivity of C . Hence the desired equality follows from Lemma 2.3.

If C is a Cartan factor of type 5 or 6, then it is finite-dimensional and $A \bar{\otimes} C$ is defined to be the injective tensor product. Further, by [12, 3.2] or [57, p. 24] we get $(A_* \hat{\otimes}_\pi C_*)^* = B(A_*, C)$ which coincides with the injective tensor product $A \hat{\otimes}_\varepsilon C$, as C has finite dimension. \square

Lemma 4.9. *Let $(M_\gamma)_{\gamma \in \Gamma}$ be an indexed family of JBW^* -triples, and let us denote*

$$M = \left(\bigoplus_{\gamma \in \Gamma} M_\gamma \right)_{\ell_\infty}.$$

$$M_\sigma = \left\{ (x_\gamma)_{\gamma \in \Gamma} \in M : x_\gamma \in (M_\gamma)_\sigma \text{ for } \gamma \in \Gamma \text{ \& \{ } \gamma \in \Gamma : x_\gamma \neq 0 \} \text{ is countable} \right\}.$$

Proof. This follows easily if we observe that $e = (e_\gamma)_{\gamma \in \Gamma} \in M$ is a tripotent if and only if e_γ is a tripotent for each γ and, moreover, e is σ -finite if and only if each e_γ is σ -finite and only countably many e_γ are nonzero. \square

Proposition 4.10. *Let A be a commutative von Neumann algebra and C a reflexive Cartan factor. (This applies, in particular, to Cartan factors of types 4, 5 and 6.) Let $M = A \bar{\otimes} C$. Then M_σ is a 1-norming Σ -subspace of $M = (M_*)^*$, and hence*

M_* is 1-Plichko. Furthermore, M_* is weakly Lindelöf determined if and only if A is σ -finite. In such a case M_* is even weakly compactly generated.

Proof. If A is a commutative von Neumann algebra, by [60, Theorem III.1.18] it can be represented as $L^\infty(\Omega, \mu)$, where Ω is a locally compact space and μ a positive Radon measure on Ω . In fact, Ω is the topological sum of a family of compact spaces $(K_\gamma)_{\gamma \in \Gamma}$. Then the predual of A is identified with

$$L^1(\Omega, \mu) = \left(\bigoplus_{\gamma \in \Gamma} L^1(K_\gamma, \mu|_{K_\gamma}) \right)_{\ell_1}.$$

Since

$$(A \overline{\otimes} C)_* = A_* \widehat{\otimes}_\pi C_* = L^1(\mu, C_*),$$

we can use Theorem 4.6. To complete the proof it is enough to show that $S = M_\sigma$, where S is the Σ -subspace provided by Theorem 4.6. Since

$$M = \left(\bigoplus_{\gamma \in \Gamma} L^\infty(K_\gamma, \mu|_{K_\gamma}, C) \right)_{\ell_\infty},$$

due to Lemma 4.9, it is enough to show that $L^\infty(\mu, C)$ is σ -finite whenever μ is finite. But, in this case, its predual, $L^1(\mu, C_*)$, is weakly compactly generated by Proposition 4.7, thus $L^\infty(\mu, C)$ is σ -finite by Theorem 4.6. \square

Proof of Theorem 4.1. We have already mentioned that it is enough to show (10). Let M be a JBW*-triple and consider the decomposition (8). By Propositions 4.5 and 4.10 each summand fulfills (10). Further, Lemma 4.9 and [39, Lemma 4.34] yield the validity of (10) for M . \square

In passing we remark that from Theorem 4.1 (and the general facts on Plichko spaces) we have that M_σ is norm-closed and even weak*-countably closed; it is additionally weak*-closed if and only if M is σ -finite.

5. STRUCTURE OF THE SPACE M_σ

In the previous section we proved that, for any JBW*-triple M , M_σ is a 1-norming Σ -subspace of $M = (M_*)^*$. If M is σ -finite, it is the only 1-norming Σ -subspace and coincides with the whole M . If M is not σ -finite, there may be plenty of different 1-norming Σ -subspaces (cf. [39, Example 6.9]). However, M_σ is the only canonical 1-norming Σ -subspace. What we mean by this statement is in the content of the following theorem.

Theorem 5.1. *Let M be a JBW*-triple. Then M_σ is a norm-closed inner ideal in M . Moreover, it is the only 1-norming Σ -subspace which is also an inner ideal.*

The theorem will be proved at the end of this section.

The following technical result provides a characterization of σ -finite tripotents which is required later. We recall that, given a tripotent u in a JBW*-triple M , there exists a complete tripotent $w \in M$ such that $u \leq w$ (see [34, Lemma 3.12(1)]).

Proposition 5.2. *Let u be a tripotent in a JBW*-triple M . The following statements are equivalent:*

- (a) u is σ -finite;

- (b) *There exist a σ -finite tripotent v and a complete tripotent w in M such that $v \leq w$ and $(w - v) \perp u$.*

Proof. The implication (a) \Rightarrow (b) is clear with $v = u$.

(b) \Rightarrow (a) Suppose there exist a σ -finite tripotent v and a complete tripotent w in M such that $v \leq w$ and $(w - v) \perp u$. Writing $w = v + (w - v)$ and using successively the orthogonality of $w - v$ to u and to v we obtain $\{w, w, u\} = \{w, v, u\} = \{v, v, u\}$, and hence $L(w, w)u = L(v, v)u$, and similarly $\{w, u, w\} = \{v, u, v\}$. Since $w - v \perp M_2(v) \ni \{v, u, v\}$, it follows that $P_2(w)(u) = Q(w)^2(u) = \{w, \{v, u, v\}, w\} = \{v, \{v, u, v\}, v\} = P_2(v)(u)$. Therefore, $P_2(w)(u) = P_2(v)(u)$ and $P_1(w)(u) = 2L(w, w)(u) - 2P_2(w)(u) = P_1(v)(u)$.

The completeness of w assures that $u = P_2(w)(u) + P_1(w)(u) = P_2(v)(u) + P_1(v)(u)$ lies in $M_2(v) \oplus M_1(v)$.

We shall show now that u is σ -finite. Arguing by contradiction, assume there is an uncountable family $(u_j)_{j \in \Gamma}$ of mutually orthogonal non-zero tripotents in M with $u_j \leq u$ for every j (see [19, §3]). Since $u_j \in M_2(u)$ for every j and $u \perp (w - v)$, it follows that $u_j \perp (w - v)$ for every $j \in \Gamma$. Arguing as above we obtain $u_j \in M_2(v) \oplus M_1(v)$, for every $j \in \Gamma$.

Having in mind that v is σ -finite, we can find a norm one functional $\phi_v \in M_*$ whose support tripotent is v (see [19, Theorem 3.2]). By Lemma 3.3, ϕ_v gives rise to a norm $\|\cdot\|_{\phi_v}$ on $M_2(v) \oplus M_1(v)$ defined by $\|x\|_{\phi_v} = (\phi_v\{x, x, v\})^{1/2}$ ($x \in M_2(v) \oplus M_1(v)$). As u_j is a non-zero element in $M_2(v) \oplus M_1(v)$ by the preceding paragraph, we obtain

$$\phi_v\{u_j, u_j, v\} = \|u_j\|^2 > 0.$$

Therefore, there exists a positive constant Θ and an uncountable subset $\Gamma' \subseteq \Gamma$ such that $\phi_v\{u_j, u_j, v\} > \Theta$ for all $j \in \Gamma'$. Thus, for each natural m we can find $j_1 \neq j_2 \neq \dots \neq j_m \in \Gamma'$. Since the elements u_{j_1}, \dots, u_{j_m} are mutually orthogonal, we get

$$\begin{aligned} 1 &= \left\| \sum_{k=1}^m u_{j_k} \right\|^2 \geq \left\| \sum_{k=1}^m u_{j_k} \right\|_{\phi_v}^2 = \phi_v \left\{ \sum_{k=1}^m u_{j_k}, \sum_{k=1}^m u_{j_k}, v \right\} \\ &= \sum_{k=1}^m \phi_v \{u_{j_k}, u_{j_k}, v\} > m\Theta, \end{aligned}$$

which is impossible. \square

To prove that M_σ is an inner ideal, we need another representation of M . To this end fix a complete tripotent $e \in M$. Applying Theorem 3.4(ii) in [19] we can find a family $(e_\lambda)_{\lambda \in \Lambda}$ of mutually orthogonal σ -finite tripotents in M satisfying $e = \sum_{\lambda \in \Lambda} e_\lambda$. For each $x \in M$ let us define

$$\Lambda_x := \{\lambda \in \Lambda : L(e_\lambda, e_\lambda)(x) \neq 0\}.$$

Proposition 5.3. *In the conditions above,*

$$M_\sigma = \{x \in M : \Lambda_x \text{ is countable}\},$$

and M_σ is a norm-closed inner ideal of M .

Proof. Denote the set on the right-hand side by M'_σ . By the linearity of the Jordan product in the third variable it follows that M'_σ is a linear subspace. To show that it is an inner ideal, take $x, z \in M'_\sigma$ and $y \in M$. For each $\lambda \in \Lambda \setminus (\Lambda_x \cup \Lambda_z)$, we deduce via Jordan identity, that

$$\begin{aligned} L(e_\lambda, e_\lambda) \{x, y, z\} &= \{L(e_\lambda, e_\lambda)x, y, z\} - \{x, L(e_\lambda, e_\lambda)y, z\} + \{x, y, L(e_\lambda, e_\lambda)z\} \\ &= -\{x, L(e_\lambda, e_\lambda)y, z\}. \end{aligned}$$

Moreover, since $L(e_\lambda, e_\lambda)x = L(e_\lambda, e_\lambda)z = 0$, we get $x, z \in M_0(e_\lambda)$. Since $P_0(e_\lambda)y$ is in the 0-eigenspace of $L(e_\lambda, e_\lambda)$ we have that $L(e_\lambda, e_\lambda)(y) \in M_1(e_\lambda) \oplus M_2(e_\lambda)$ and hence $\{x, L(e_\lambda, e_\lambda)(y), z\} = 0$ by Peirce arithmetic. We have shown that $\Lambda_{\{x, y, z\}} \subseteq \Lambda_x \cup \Lambda_z$, and thus $\Lambda_{\{x, y, z\}}$ is countable, which proves that $\{x, y, z\} \in M'_\sigma$ and hence M'_σ is an inner ideal of M .

We continue by showing that $M_\sigma \subset M'_\sigma$. We shall first prove that M'_σ contains all σ -finite tripotents in M . Let u be a σ -finite tripotent in M . We want to show that the set Λ_u is countable. We assume, on the contrary, that Λ_u is uncountable. Let $\phi_u \in M_*$ be a norm one functional whose support tripotent is u . For every $\lambda \in \Lambda_u$, we have that $e_\lambda \notin M_0(u)$ because otherwise we would have $L(e_\lambda, e_\lambda)(u) = 0$. Consequently, as in the proof of Proposition 5.2, we deduce that $\phi_u\{e_\lambda, e_\lambda, u\} > 0$. We can thus find a positive constant Θ and an uncountable subset $\Lambda'_u \subseteq \Lambda_u$ such that $\phi_u\{e_\lambda, e_\lambda, u\} > \Theta$ for all $\lambda \in \Lambda'_u$. As before, for each natural m we can find $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_m \in \Lambda'_u$. Then, applying the orthogonality of the elements e_{λ_j} we get

$$\begin{aligned} 1 &= \left\| \sum_{j=1}^m e_{\lambda_j} \right\|^2 \geq \left\| \sum_{j=1}^m e_{\lambda_j} \right\|_{\phi_u}^2 = \phi_u \left\{ \sum_{j=1}^m e_{\lambda_j}, \sum_{j=1}^m e_{\lambda_j}, u \right\} \\ &= \sum_{j=1}^m \phi_u \{e_{\lambda_j}, e_{\lambda_j}, u\} > m\Theta, \end{aligned}$$

which gives a contradiction. This proves that Λ_u is countable, and hence $u \in M'_\sigma$.

Let us now assume that x is any element of M_σ . Then its range tripotent, $r(x)$, is σ -finite and hence $r(x) \in M'_\sigma$ by the previous paragraph. Since $x \in M_2(r(x))$ is a positive and hence self-adjoint element, we have $\{r(x), x, r(x)\} = x$ and hence $x \in M'_\sigma$ as M'_σ is an inner ideal. This shows that $M_\sigma \subset M'_\sigma$.

Conversely, let $x \in M'_\sigma$. In this case the set Λ_x is countable. The tripotent $u = w^* - \sum_{\lambda \in \Lambda_x} e_\lambda$ is σ -finite in M , $e = u + v$, where $v = w^* - \sum_{\lambda \in \Lambda \setminus \Lambda_x} e_\lambda$ is another tripotent in M with $u \perp v$. Since $\{e_\lambda, e_\lambda, x\} = 0$ for all $\lambda \in \Lambda \setminus \Lambda_x$, it follows from the separate weak*-continuity of the triple product of M that $\{v, v, x\} = 0$, that is, $x \in M_0(v)$. Hence also $r(x) \in M_0(v)$ (as $M_0(v)$ is a JBW*-subtriple of M). It follows that $r(x) \perp v$ and hence $r(x)$ is σ -finite by Proposition 5.2.

We finally observe that, by Theorem 4.1, M_σ is a Σ -subspace and hence it is norm-closed (cf. Lemma 2.5(i)). This completes the proof. \square

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.1. M_σ is a norm-closed inner ideal by Proposition 5.3. Let us prove the uniqueness.

Let I be an inner ideal which is a 1-norming Σ -subspace. We will show that I contains all sigma-finite tripotents. Let $e \in M$ be a sigma-finite tripotent, $\phi \in M_*$ a normal functional of norm 1 such that e is the support tripotent of ϕ . By Lemma 2.5(v) there is $x \in I$ of norm 1 with $\phi(x) = 1$. Further, we get $r(x) \in I$. Indeed, $r(x)$ is contained in the weak*-closure of the JB*-subtriple of M generated by x . Since this subtriple is norm-separable, we get $r(x) \in I$ by Lemma 2.5(i).

In order to show $e \in I$ it is enough to show that $e \leq r(x)$. By (4) it is enough to prove that $\phi(r(x)) = 1$. Proposition 2.5 in [50] assures that $\phi(x^{[\frac{1}{2n+1}]}) = \phi(x)^{[\frac{1}{2n+1}]} = 1$, for all natural n . Since ϕ is a normal functional and $(x^{[\frac{1}{2n+1}]}) \rightarrow r(x)$ in the weak* topology of M , it follows that $\phi(r(x)) = 1$, as we desired.

Now, if $z \in M_\sigma$ is arbitrary, then there is a σ -finite tripotent $f \in M$ with $z \in M_2(f)$. By the above we have $f \in I$. Since I is an inner ideal, we conclude that $M_2(f) \subset I$, and hence $z \in X$.

Therefore, $M_\sigma \subset I$. Lemma 2.5(iii) now shows that $M_\sigma = I$. \square

Remark 5.4. *It is possible to give a shorter proof of the fact that the predual of a JBW*-triple is 1-Plichko by using the main result of [5] at the cost of applying elementary submodels theory. However, this alternative argument does not yield M_σ as a concrete description of a Σ -subspace. We shall only sketch this variant:*

First, it is not too difficult to modify the decomposition (8) by writing

$$(11) \quad M = \left(\bigoplus_{j \in \mathcal{I}} A_j \overline{\otimes} G_j \right)_{\ell_\infty} \oplus_{\ell_\infty} N \oplus_{\ell_\infty} pV,$$

where each A_j is a commutative von Neumann algebra, each G_j is a finite dimensional Cartan factor, p is a projection in a von Neumann algebra V , and N is a JBW-algebra.*

Second, an almost word-by-word adaptation of the proof of [4, Theorem 1.1] shows that the predual of pV is 1-Plichko (compare Proposition 4.5). So is the predual of N by the main result of [5]. Finally, the summands $A_j \overline{\otimes} G_j$ are seen to have 1-Plichko predual as in the proof of 4.6 (or by an easier argument using the finite dimensionality of C_j), and the stability of 1-Plichko spaces by ℓ_1 -sums ([39, Theorem 4.31(iii)] or Lemma 4.9) allows us to conclude.

6. THE CASE OF REAL JBW*-TRIPLES

Introduced by J.M. Isidro, W. Kaup, and A. Rodríguez (see [37]), *real JB*-triples* are, by definition, the closed real subtriples of JB*-triples. Every complex JB*-triple is a real JB*-triple when we consider the underlying real Banach structure. Real and complex C*-algebras belong to the class of real JB*-triples. An equivalent reformulation asserts that real JB*-triples are in one-to-one correspondence with the real forms of JB*-triples. More precisely, for each real JB*-triple E there exist a (complex) JB*-triple E_c and a period-2 conjugate-linear isometry (and hence a conjugate-linear triple isomorphism) $\tau : E_c \rightarrow E_c$ such that $E = \{b \in E_c : \tau(b) = b\}$. The JB*-triple E_c identifies with the complexification of E (see [37, Proposition 2.2] or [10, Proposition 4.2.54]). In particular, every JB-algebra (and hence the self-adjoint part, A_{sa} of every C*-algebra A) is a real JB*-triple.

Henceforth, for each complex Banach space X , the symbol $X_{\mathbb{R}}$ will denote the underlying real Banach space.

In the conditions above we can consider another period-2 conjugate-linear isometry $\tau^\sharp : E_c^* \rightarrow E_c^*$ defined by

$$\tau^\sharp(\varphi)(z) := \overline{\varphi(\tau(z))} \quad (\varphi \in E_c^*).$$

It is further known that the operator

$$(E_c^*)^{\tau^\sharp} \rightarrow (E_c^\tau)^*, \quad \varphi \mapsto \varphi|_E$$

is an isometric real-linear bijection, where $(E_c^*)^{\tau^\sharp} := \{\varphi \in E_c^* : \tau^\sharp(\varphi) = \varphi\}$.

A real JBW*-triple is a real JB*-triple which is also a dual Banach space ([37, Definition 4.1] and [48, Theorem 2.11]). It is known that every real JBW*-triple admits a unique (isometric) predual and its triple product is separately weak*-continuous (see [48, Proposition 2.3 and Theorem 2.11]). Actually, by the just quoted results, given a real JBW*-triple N there exists a JBW*-triple M and a weak*-to-weak* continuous period-2 conjugate-linear isometry $\tau : M \rightarrow M$ such that $N = M^\tau$. The mapping τ^\sharp maps M_* into itself, and hence we can identify $(M_*)^{\tau^\sharp}$ with $N_* = (M^\tau)_*$. We can also consider a weak*-continuous real-linear bicontractive projection $P = \frac{1}{2}(Id + \tau)$ of M onto $N = M^\tau$, and a bicontractive real-linear projection of M_* onto N_* defined by $Q = \frac{1}{2}(Id + \tau^\sharp)$. From now on, N , M , τ , P , and Q will have the meaning explained in this paragraph.

Due to the general lack for real JBW*-triples of the kind of structure results established by Horn and Neher for JBW*-triples in [35, 36], the proofs given in section 4 cannot be applied for real JBW*-triples. Despite of the limitations appearing in the real setting, we shall see how the tools in previous section can be applied to prove that preduals of real JBW*-triples are 1-Plichko spaces too.

We shall need to extend the concept of σ -finite tripotents to the setting of real JBW*-triples. The notions of tripotents, Peirce projections, Peirce decomposition are perfectly transferred to the real setting. The relations of orthogonality and order also make sense in the set of tripotents in N (cf. [37, 48]). Furthermore, for each tripotent e in N , $Q(e)$ induces a decomposition of N into \mathbb{R} -linear subspaces satisfying

$$N = N^1(e) \oplus N^0(e) \oplus N^{-1}(e),$$

where $N^k(e) := \{x \in N : Q(e)x = kx\}$,

$$N_2(e) = N^1(e) \oplus N^{-1}(e), \quad N^0(e) = N_1(e) \oplus N_0(e),$$

$$\{N^j(e), N^k(e), N^\ell(e)\} \subset N^{jkl}(e) \text{ if } jkl \neq 0, \quad j, k, \ell \in \{0, \pm 1\}, \text{ and zero otherwise.}$$

The natural projection of N onto $N^k(e)$ is denoted by $P^k(e)$. It is also known that $P^1(e)$, $P^{-1}(e)$, and $P^0(e)$ are all weak*-continuous. The subspace $N^1(e)$ is a weak*-closed Jordan subalgebra of the JBW-algebra $(M_2(e))_{sa}$, and hence $N^1(e)$ is a JBW-algebra.

Given a normal functional $\phi \in N_*$, there exists a normal functional $\varphi \in M_*$ satisfying $\tau^\sharp(\varphi) = \varphi$ and $\varphi|_N = \phi$. Let $e(\varphi)$ be the support tripotent of φ in M . Since $1 = \varphi(e(\varphi)) = \overline{\varphi(\tau(e(\varphi)))} = \varphi(\tau(e(\varphi)))$, we deduce that $\tau(e(\varphi)) \geq e(\varphi)$. Applying that τ is a triple homomorphism, we get $e(\varphi) = \tau^2(e(\varphi)) \geq \tau(e(\varphi)) \geq e(\varphi)$, which proves that $e(\varphi) = \tau(e(\varphi)) \in N$. That is, the support tripotent of a τ^\sharp -symmetric normal functional φ in M_* is τ -symmetric. The tripotent $e(\varphi)$ is called the support tripotent of ϕ in N , and it is denoted by $e(\phi)$. It is known

that $\phi = \phi P^1(e(\phi))$ and $\phi|_{N^1(e(\phi))}$ is a faithful positive normal functional on the JBW-algebra $N^1(e(\phi))$ (compare [52, Lemma 2.7]).

As in the complex setting, a tripotent e in N is called σ -finite if e does not majorize an uncountable orthogonal subset of tripotents in N . The real JBW*-triple N is called σ -finite if every tripotent in N is σ -finite.

Proposition 6.1. *In the setting fixed for this section, let e be a tripotent in N . The following are equivalent:*

- (a) e is σ -finite in N ;
- (b) e is σ -finite in M ;
- (c) e is the support tripotent of a normal functional ϕ in N_* ;
- (d) e is the support tripotent of a τ^\sharp -symmetric normal functional φ in M_* .

Consequently, for

$$N_\sigma := \{x \in N : \text{there exists a } \sigma\text{-finite tripotent } e \text{ in } N \text{ with } \{e, e, x\} = x\}$$

we have

$$N_\sigma = \{x \in M_\sigma : \tau(x) = x\} = N \cap M_\sigma,$$

and the following are equivalent:

- (i) M is σ -finite (i.e. $M_\sigma = M$);
- (ii) N is σ -finite (i.e. $N_\sigma = N$);
- (iii) N contains a complete σ -finite tripotent.

Proof. The implication (b) \Rightarrow (a) and the equivalence (c) \Leftrightarrow (d) are clear. The implication (d) \Rightarrow (b) follows from [19, Theorem 3.2]. To see (a) \Rightarrow (d), let us assume that e is σ -finite in N . Clearly e is the unit in the JBW-algebra $N^1(e)$, and since every family of mutually orthogonal projections in this algebra is a family of mutually orthogonal tripotents in N majorized by e , we deduce that e is a σ -finite projection in $N^1(e)$. Theorem 4.6 in [14] assures the existence of a faithful normal state ϕ in $(N^1(e))_*$. By a slight abuse of notation, the symbol ϕ will also denote the functional $\phi P^1(e)$. Clearly $\phi \in N_*$ and $\phi|_{N^1(e)}$ is a faithful normal state.

By the arguments above, there exists a τ^\sharp -symmetric normal functional φ in M_* such that $\varphi|_N = \phi$. Let $e(\varphi)$ be the support tripotent of φ in M . We have also commented before this proposition that $\tau(e(\varphi)) = e(\varphi)$ (i.e. $e(\varphi) \in N$) because ϕ is τ^\sharp -symmetric. Since $\varphi(e) = \phi(e) = 1$, we deduce that $e \geq e(\varphi)$. Therefore $e(\varphi)$ is a projection in the JBW-algebra $N^1(e)$. Furthermore, $\phi(e(\varphi)) = \varphi(e(\varphi)) = 1$ and the faithfulness of $\phi|_{N^1(e)}$ show that $e = e(\varphi)$. This proves the equivalence of (a), (b), (c) and (d). The equality $N_\sigma = N \cap M_\sigma$ is clear from the first statement.

Since a complete tripotent in N is a complete tripotent in M , the rest of the statement follows from the previous equivalences and [19, Theorem 4.4]. \square

We can prove now our main result for preduals of real JBW*-triples.

Theorem 6.2. *The predual of any real JBW*-triple N is a 1-Plichko space. Moreover, N_* is weakly Lindelöf determined if and only if N is σ -finite. In the latter case N_* is even weakly compactly generated.*

Proof. We keep the notation fixed for this section with N , M and τ as above. There exists a canonical isometric identification of $M_\mathbb{R}$ with $((M_*)_\mathbb{R})^*$, where any $x \in M_\mathbb{R}$ acts on $(M_*)_\mathbb{R}$ by the assignment $\omega \mapsto \text{Re } \omega(x)$ ($\omega \in (M_*)_\mathbb{R}$). Thus $(M_*)_\mathbb{R}$ is a real

1-Plichko space and M_σ is again a 1-norming σ -subspace by Theorem 4.1 and [41, Proposition 3.4].

In view of Lemma 2.6 to prove that the predual of N is 1-Plichko, it is enough to show that $\mathcal{B}_N \cap M_\sigma$ is weak*-dense in \mathcal{B}_N . Since M_σ is a 1-norming subspace we can easily see that \mathcal{B}_{M_σ} is weak*-dense in \mathcal{B}_M . Take an element $a \in \mathcal{B}_N \subset \mathcal{B}_M$. Then there exists a net $(a_\lambda) \subset \mathcal{B}_{M_\sigma}$ converging to a in the weak*-topology of M . Since τ is weak*-continuous and M_σ is a norm-closed τ -invariant subspace of M , we can easily see that $(\frac{a_\lambda + \tau(a_\lambda)}{2}) \rightarrow a$ in the weak*-topology of M , where $(\frac{a_\lambda + \tau(a_\lambda)}{2}) \subset \mathcal{B}_{N_\sigma} = \mathcal{B}_N \cap M_\sigma$, which proves the desired weak*-density.

For the last statement, we observe that N is σ -finite if and only if M is (see Proposition 6.1), and hence the desired equivalence follows from Theorem 4.1 and the results presented in sections 4 and 6. We also note that N σ -finite implies M σ -finite implies M_* WCG implies N_* WCG, being a complemented subspace. \square

We can rediscover the following two results in [4] and [5] as corollaries of our last theorem.

Corollary 6.3. [4, Theorem 1.4] *Let W be a von Neumann algebra. Then the predual, $(W_{sa})_*$, of the self-adjoint part, W_{sa} , of W is a 1-Plichko space. Moreover, $(W_{sa})_*$ is weakly Lindelöf determined if and only if W is σ -finite. In the latter case W_* and $(W_{sa})_*$ are even weakly compactly generated.* \square

Corollary 6.4. [5, Theorem 1.1] *The predual of any JBW-algebra J is 1-Plichko. Moreover, J_* is weakly Lindelöf determined if and only if J is σ -finite. In the latter case J_* is even weakly compactly generated.* \square

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